Allan variance: variations and application to metrology gauge data

METGAUGE MEMO ALLAN-VAR

RES

9 Feb 99

1 Introduction

Many of the performance requirements of the SIM mission can be summarized in a σ–τ plot, in which the allowable error σ is expressed as a function of the averaging time τ. It has been suggested that the Allan variance σ_A^2 is the appropriate quantity for these plots; that is, the Allan deviation $\sigma(\tau) = \sqrt{\sigma_A^2(\tau)}$ is a good estimator of the error in an averaged measurement of the noise-equivalent pathlength x. Indeed that statistical tool—which was originally developed to characterize the frequency stability of clocks—can be applied to the type of data generated in SIM research and measurements. The purpose of this note is to define the particular form of Allan variance that we use as a metric for characterizing performance of the Metgauge experiments.

2 Time-domain definition

For concreteness, consider the output of an experiment that generates a data stream representing a displacement x (such as an interferometrically measured pathlength) as a function of time. Denote the k'th average of x over duration τ by $\bar{x}_k$. Assume that the data in hand is of duration several τ, so a series of adjacent non-overlapping averages can be defined, as in Figure 1. We are interested in estimating σ, the expected error in x when averaged over time τ. The following definition ([CL90], [LA84], [Ega88], [Spe99], [PKK91]) of Allan variance applies

$$\sigma_{A_x}^2 = \frac{1}{m}\sum_{k=1}^{m}\frac{(\bar{x}_{k+1}-\bar{x}_k)^2}{2}. \tag{1}$$

That is, the Allan variance is one half of the mean-square of the differences of successive averages. For comparison, the everyday variance of the τ-duration mean is

$$\sigma_x^2 = \frac{1}{m}\sum_{k=1}^{m}(\bar{x}_{k+1}-\bar{x})^2, \tag{2}$$

where $\bar{x}$ is the average over the whole data set. In the SIM mission, stability between adjacent averiging periods (Equation 1) is usually more relevant than the stability with respect to a global mean (Equation 2); therefore we adopt the former as the metric for estimating errors in the metrology gauge.

The Allan deviation is defined as the square root of the Allan variance.

3 Variations and data example

Note that some authors ([Wal94], [Gre97]) assume that the raw data is phase error, and that the quantity of interest is frequency error. This introduces a "second difference" complication to the computation that is relevant to the problem of inferring the frequency stability of clocks from measurements of phase stability, but is irrelevant to our purposes. In particular, it removes the error from a constant slope, which is often the quantity of interest where thermally driven

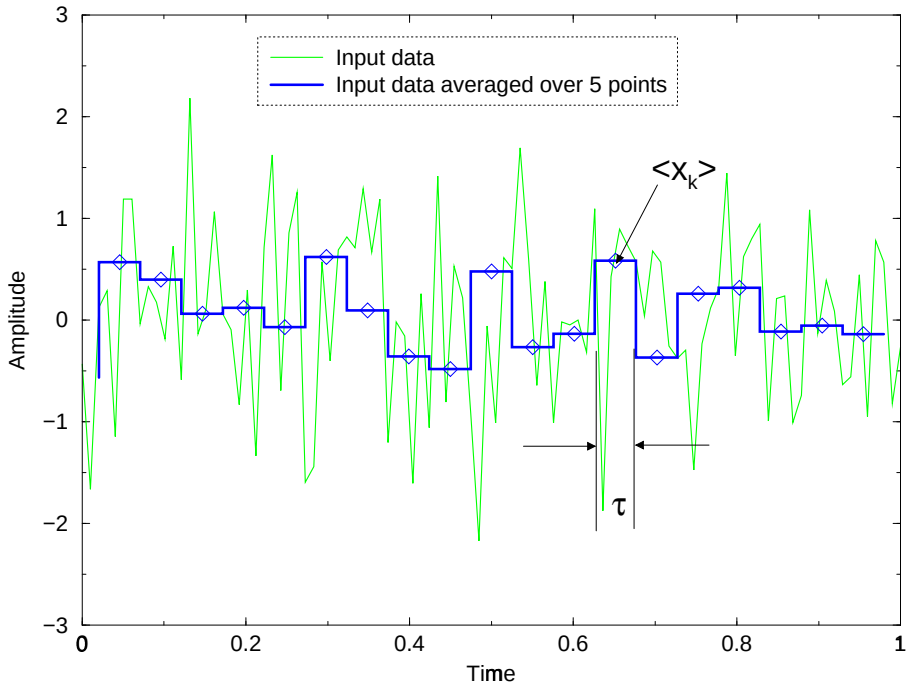

Figure 1: Averaging of input data over successive time intervals τ. In this illustration the noise is normally distributed ("white") and the averaging interval is $\tau = 5$ sample points.The horizontal step heights steps correspond to the $\bar{x}_k$ of Equation 1.

drifts are important. The comparison of the correct Allan deviation and the phase-based "Second Difference" Allan deviation in Figure 3 shows that the second difference formulation underestimates the error at large averaging time.

Another twist is the introduction of the Modified Allan variance, in which the straightforward averages of Equation 1 are replaced by an average of sliding averages ([LA84]). The purpose of this modification is to make the inference of Power Spectral Density (PSD) from Allan variance more precise. However this inversion process is at best approximate, and since we always have the PSD available directly there seems little benefit to using the Modified Allan variance. ([LA84] even advises against the use of Mod σ_A^2, based on an analysis of its statistical properties.)

The Allan variance and its friends can also be computed starting with the PSD. In general,

$$\sigma^2 = \int_0^\infty S(f) h^2(f)\, df, \tag{3}$$

where $S(f)$ is the PSD of the time-series, σ stands for the three types of variance, and the form of the weighting factor $h^2(f)$ determines which σ is calculated. The weighting factors for variance and Allan variance are given in Table 1. For reference, the weighting factor for the Modified Allan variance is

$$h^2(f) = \frac{2\sin^4(\pi f \tau)}{n^2 \pi^2 \tau^2 f^2} \left[n + \sum_{k=1}^{n-1} (n-k) \cos(\frac{k}{n} 2\pi f \tau) \right] \tag{4}$$

where there are n sliding average segments per averaging time τ. This reduces to the form in the second row of Table 1 when $n = 1$. A comparison of these methods applied to the same data set is shown in Figure 3. Evidently, the PSD-based calculations become inaccurate for large τ.

References

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Metric		Time domain weighting	$h^2(f)$	Reference
Variance	σ	Fig. 2a	$2\left(\frac{\sin \pi f \tau}{\pi f \tau}\right)^2$	[Pap77], eq. 10–34
Allan Variance	σ_A	Fig. 2b	$2\left(\frac{\sin^2 \pi f \tau}{\pi f \tau}\right)^2$	[PKK91]
Modified Allan Variance	$\mathrm{Mod}\,\sigma_A$	Fig. 2c	Equation 4	[LA84]

Table 1: Weighting factors in time- and frequency-domain for computation of variance, Allan variance, and Modified Allan variance.

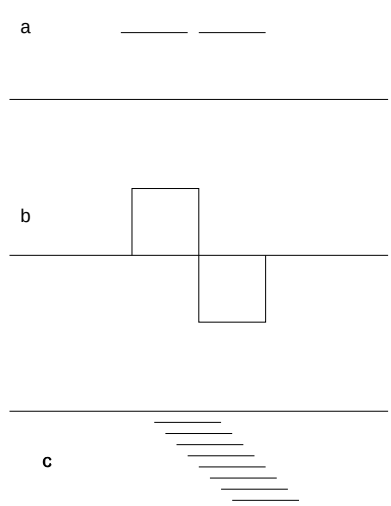

Figure 2: Schematic of the algorithm for calculating variance *(a)*, Allan variance *(b)*, and Modified Allan variance *(c)*. The horizontal bars are each of duration equal to the averaging time τ; the staggered sequence of bars in *c* is meant to indicate a sequence of sliding averages, after [LA84].

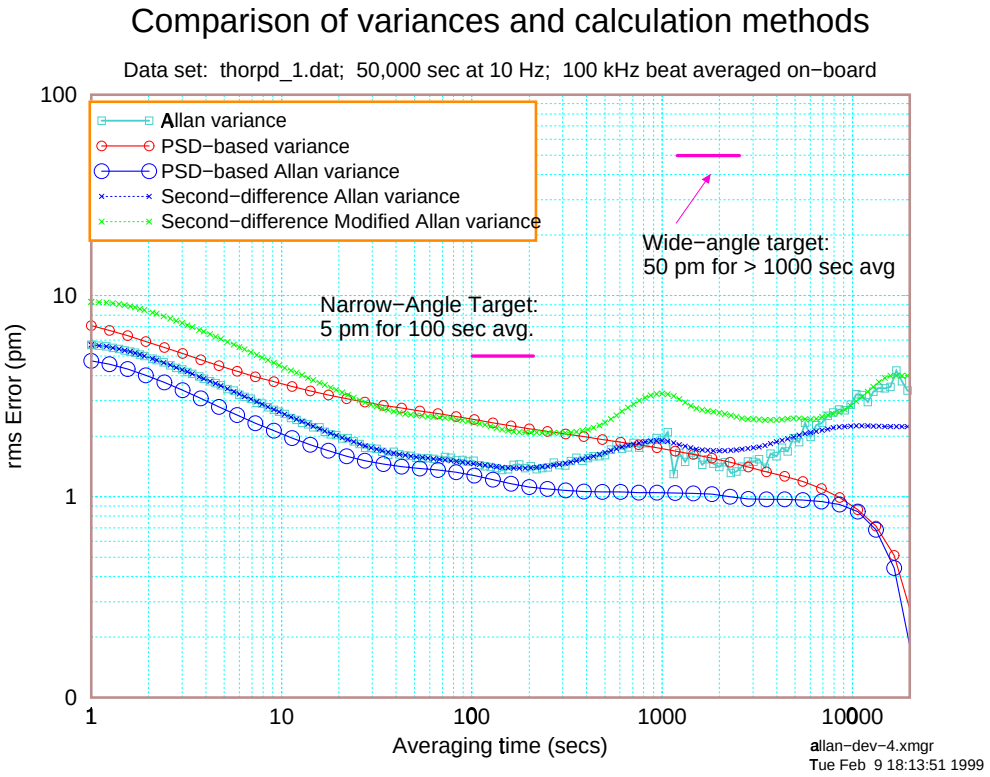

Figure 3: Comparison of different methods of calculating various types of variance; based on a common 50,000-second data set from a benchtop beam launcher reference signal.

[Ega88] Willaim F. Egan. An efficient algorithm to compute Allan variance from spectral density. *IEEE Transactions on Instrumentation and Measurement*, 37(2):240–244, June 1988.

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[LA84] Paul Lesage and Theophane Ayi. Characterization of frequency stability: analysis of the modified Allan variance and properties of its estimate. *IEEE Transactions on Instrumentation and Measurement*, IM-33(4):332–336, December 1984.

[Pap77] Athanasios Papoulis. *Signal Analysis*. McGraw-Hill, 1977.

[PKK91] W.V. Prestwich, T.J. Kennett, and F.W. Kus. The statistical properties of Allan variance. *Can. J. Phys.*, 69:1405–1415, 1991.

[Spe99] Robert E. Spero. On-obard averaging: What's it good for? Technical report, JPL Metgauge memo, February 1999. Web accessable at http://huey.jpl.nasa.gov/∼respero/psd-avg.pdf.

[Wal94] Todd Walter. Characterizing frequency stability: a continuous power-law model with discrete sampling. *IEEE Transactions on Instrumentation and Measurement*, 43(1):69–79, February 1994.

Several of the above references have recently become available on-line to NASA employees, at
http://www.sti.nasa.gov/nasaonly/ielfram1.htm